Getting to the Bottom of Noether's Theorem

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It’s sometimes said Noether showed *symmetries give conservation laws*. But this is only true under some assumptions: for example, that the equations of motion come from a Lagrangian.

For which types of physical theories do symmetries give conservation laws?

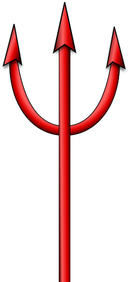
What are we assuming about the world, if we assume it’s described by theories of this type?

It’s hard to get to the bottom of these questions, but let’s try.

We can prove versions of Noether's theorem in many frameworks, including classical, quantum, and stochastic.

In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics. — *Hermann Weyl*

I will study this subject *algebraically*.



A version of Noether's theorem holds almost tautologously if observables form a Poisson algebra.

A **Poisson algebra** is a real vector space A equipped with a multiplication making A into a commutative algebra:

$$a(bc) = (ab)c \qquad ab = ba$$

$$a(\beta b + \gamma c) = \beta ab + \gamma ac$$

together with a **Poisson bracket** making A into a Lie algebra:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\} \qquad \{a, b\} = -\{b, a\}$$

$$\{a, \beta b + \gamma c\} = \beta\{a, b\} + \gamma\{a, c\}$$

and obeying the **Leibniz law**:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

In classical mechanics a Poisson algebra A serves as our algebra of observables. Assume for any $a \in A$ there is a unique one-parameter group of maps

$$F_t^a \colon A \to A \qquad (t \in \mathbb{R})$$

obeying

$$\frac{d}{dt} F_t^a(b) = \{a, F_t^a(b)\}$$

Here **one-parameter group** means

$$F_0^a(b) = b$$

$$F_s^a(F_t^a(b)) = F_{s+t}^a(b) \qquad \forall s, t \in \mathbb{R}$$

These assumption holds in all ‘nice’ cases. (Consult the demon of analysis.)

Then we can check that the maps F_t^a act as symmetries of our Poisson algebra:

$$F_t^a(\beta b + \gamma c) = \beta F_t^a(b) + \gamma F_t^a(c)$$

$$F_t^a(bc) = F_t^a(b) F_t^a(c)$$

$$F_t^a(\{b, c\}) = \left\{F_t^a(b), F_t^a(c)\right\}$$

Suppose we take $a \in A$ as our Hamiltonian, so that $F_t^a \colon A \to A$ describes the time evolution of observables.

We say $b \in A$ is a **conserved quantity** if

$$F_t^a(b) = b \qquad \forall t \in \mathbb{R}$$

We say $b \in A$ **generates symmetries of the Hamiltonian** if

$$F_t^b(a) = a \qquad \forall t \in \mathbb{R}$$

These are equivalent! And the proof is very pretty.

$$F_t^a(b) = b \qquad \forall t \in \mathbb{R}$$

iff

$$\frac{d}{dt} F_t^a(b) = 0 \qquad \forall t \in \mathbb{R}$$

iff

$$\left\{a, F_t^a(b)\right\} = 0 \qquad \forall t \in \mathbb{R}$$

iff this is true at $t = 0$:

$$\{a, b\} = 0$$

By the antisymmetry of the bracket this is true iff

$$\{b, a\} = 0$$

so running the argument backwards with a, b switched we get

$$F_t^b(a) = a \qquad \forall t \in \mathbb{R}$$

So, in Poisson mechanics, observables that *generate symmetries of the Hamiltonian* are the same as *conserved quantities*. And this follows from two main ideas:

- observables generate 1-parameter transformation groups via the bracket
- the bracket is antisymmetric, so:

$$\begin{array}{ll} \{a, b\} = 0 & (b \text{ is conserved by the transformations} \\ & \quad \text{generated by } a) \\ & \Updownarrow \\ \{b, a\} = 0 & (a \text{ is conserved by the transformations} \\ & \quad \text{generated by } b) \end{array}$$

A lot of this is “just math” — more precisely, group theory!

Any Lie group G has a Lie algebra L. L has a bracket that is antisymmetric.

Any $a \in L$ gives rise to a one-parameter group of maps

$$F_t^a \colon L \to L \qquad (t \in \mathbb{R})$$

obeying

$$\frac{d}{dt} F_t^a(b) = \left[a, F_t^a(b)\right]$$

and these are symmetries of our Lie algebra:

$$F_t^a(\beta b + \gamma c) = \beta F_t^a(b) + \gamma F_t^a(c)$$

$$F_t^a([b, c]) = \left[F_t^a(b), F_t^a(c)\right]$$

The part that's *not* just group theory is this: in a Poisson algebra, the *generators of 1-parameter groups of transformations* are also the *real-valued quantities we can measure*.

We may be used to it, but it's not trivial that *symmetry generators* are also *observables*.

It's not true in some theories!

It holds in ordinary 'complex' quantum mechanics, but it fails for real and quaternionic quantum mechanics, and also for stochastic processes:

- JB and Brendan Fong, A Noether theorem for Markov processes, arXiv:1203.2035.

A Poisson algebra combines the multiplication of *observables*:

$$a(bc) = (ab)c \qquad ab = ba$$

$$a(\beta b + \gamma c) = \beta ab + \gamma ac$$

with the bracket of *symmetry generators*:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\} \qquad \{a, b\} = -\{b, a\}$$

$$\{a, \beta b + \gamma c\} = \beta\{a, b\} + \gamma\{a, c\}$$

tied together by the Leibniz law:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

It's a hybrid structure!

As we all know, the math simplifies if we take A to be a *noncommutative* but still associative algebra, and define $[a, b] = ab - ba$. Then we get the Lie algebra laws

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \qquad [a,b] = -[b,a]$$

$$[a, \beta b + \gamma c] = \beta[a,b] + \gamma[a,c]$$

and the Leibniz law

$$[a, bc] = [a,b]c + b[a,c]$$

for free! They follow from the laws of an associative algebra:

$$a(bc) = (ab)c$$

$$a(\beta b + \gamma c) = \beta ab + \gamma ac \qquad (\alpha a + \beta b)c = \alpha ac + \beta bc$$

This makes quantum mechanics much more tightly unified than classical mechanics.

But there's a catch: not all operators on Hilbert space are observables!

- real-valued *observables* are *self-adjoint*: $a^* = a$
- *symmetry generators* are *skew-adjoint*: $a^* = -a$.

We can turn self-adjoint a into skew-adjoint ia, which can generate symmetries:

$$F_t^a(b) = e^{ita}\, b\, e^{-ita}$$

This works fine, but it relies crucially on $i = \sqrt{-1}$.

Suppose $B(H)$ is the space of bounded operators on a real, complex or quaternionic Hilbert space H. Then

$$B(H) = A \oplus L$$

where

$$\begin{array}{rclcl} A & = & \{a \in B(H) :\ a^* = a\} & = & \textbf{observables} \\ L & = & \{a \in B(H) :\ a^* = -a\} & = & \textbf{symmetry generators} \end{array}$$

$B(H)$ is an associative algebra but A and L are not. L is a real Lie algebra:

$$a, b \in L \implies [a, b] := ab - ba \in L$$

A is a real Jordan algebra:

$$a, b \in A \implies a \circ b := ab + ba \in A$$

Also, the Lie algebra of symmetry generators L acts on the Jordan algebra of observables A:

$$a \in L, b \in A \implies [a, b] := ab - ba \in A$$

Any $a \in L$ gives rise to *two* one-parameter groups of maps

$$F_t^a \colon A \to A, \quad F_t^a \colon L \to L \qquad (t \in \mathbb{R})$$

both obeying

$$\frac{d}{dt} F_t^a(b) = [a, F_t^a(b)]$$

These maps preserve all the operations on A and L.

However, only in the complex case do we have a bijection

$$\phi \colon A \xrightarrow{\sim} L$$

such that

$$[a, \phi(b)] = \phi([a, b]) \qquad \forall a \in L, b \in A$$

In the complex case this map is

$$\phi(a) = ia.$$

In the real case we have no square root of -1. In the quaternionic case the different square roots of -1 fail to commute so if $a^* = a$ then usually $(ia)^* = a^* i^* = -ai \neq -ia$.

In short, the requirement that we can freely reinterpet *observables* as *symmetry generators*, and vice versa — in a way that's consistent with the action of symmetry generators on both observables and symmetry generators — is the key to Noether's theorem.

In classical mechanics this is achieved by a hybrid structure: a Poisson algebra, whose elements are both observables and symmetry generators.

In an algebraic approach to quantum theory, this requirement singles out *complex* quantum mechanics. $i = \sqrt{-1}$ turns observables into symmetry generators, and vice versa.